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Bose realization for non-canonical representations of the symplectic group $Sp(4) \supset SU(2) \times U(1)$

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Abstract
A new method is formulated for the construction of arbitrary unitary irreducible representations of the compact symplectic group $Sp(4) \sim O(5)$ in orthonormal bases which are reduced with respect to the non-canonical group chain $Sp(4) \supset SU(2) \times U(1)$. The method is based on a realization of the algebra of generators and basis states by means of a system of Bose creation and annihilation operators. As an illustration, some series of representations with multiplicities equal to, or less than, three are given in explicit algebraic form.

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1. Introduction

The canonical representations of the compact symplectic group $Sp(4) \sim O(5)$ in bases fully reduced with respect to the subgroup $SU(2) \times SU(2)$ have been completely established a long time ago by Hecht [1], Sharp and Pieper [2] and Kemmer *et al* [3], with the use of different methods. However, in physical applications, the reduction with respect to other non-canonical chains has become more natural. One of those is the chain $Sp(4) \supset SU(2) \times U(1)$, which arises, for example, in the classification of states in a nuclear j-shell as was established by Helmers [4], Flowers and Szpikowski [5] and Parikh [6]. The construction of the representations for this last scheme has been a difficult task because of the lack of one label for the complete specification of the basis states within an irreducible representation. Nevertheless, specific series of representations were derived in the past by Hecht [7, 9], Hemenger and Hecht [10], and by Szpikowski [8]. In particular all representations with multiplicities two or less were explicitly obtained in closed algebraic form. The construction of bases for generic representations was investigated by Ahmed and Sharp [11], Smirnov and Tolstoy [12] and by Szpikowski and Berej [17]. They derived explicit, though different, systems of basis vectors.

Unfortunately, the bases were non-orthogonal and the transformation to the orthogonal bases was equivalent to the diagonalization of a missing label operator, a task that can be afforded only numerically and only for specific representations. A solution to the present missing label problem was given in principle by Hecht and Elliot [13] extending the method of vector coherent states to the present problem (see also [14, 15]). They obtained a non-unitary or Dayson realization of the $so(5)$ algebra, and then derived the unitary or Holstein–Primakoff realization via a similarity transformation with the operator K of the coherent state theory. However, their method depends on the diagonalization of the K-matrix, and this has been done algebraically only for representations with maximum two-fold multiplicity, and for some particular states of arbitrary representations. In general, the diagonalization can be performed only numerically for specific, though arbitrary, representations.

The continuous interest in the theory of representations of the symplectic group $Sp(4)$ can be demonstrated by the very recent investigation of Sviratcheva *et al* [16], where fermion realizations of the $sp(4)$ algebra, as well as its q-deformations, are obtained. They considered canonical and non-canonical realizations. However, in the case of the reduction $Sp(4) \supset SU(2) \times U(1)$ considered here, they give only the realization of a particular series of symmetric representations, denoted below as $\langle q, q\rangle$ ($q = j + 1/2$ in their notation).

In the present paper, a method for the construction of arbitrary unitary irreducible representations is presented. Even though the method is appropriate for numerical computations, the algebraic expressions can be derived with relative ease, depending only on the solution of a system of linear homogeneous algebraic equations for each arbitrary series of representations $\langle q + p, q\rangle$, with q any positive integer and $p = 1, 2, \ldots$. This approach represents a different and more explicit solution to the present missing label problem, in a form that can be useful for physical applications. The method is based on a Bose realization of the representation space and the generators of the group (Jordan–Schwinger mapping), and makes a straightforward use of the $su(2)$ tensor algebra. Some examples of the derivations are presented, including the series $\langle q + 4, q\rangle$ with multiplicities equal to, or less than, 3.

Following Bincer [18], let us denote the generators of $Sp(4)$ as G^a_b, $a, b = -2, \ldots, 2$, zero excluded. They satisfy the identities $G^a_b = -\epsilon^a\epsilon^b G^{-b}_{-a}$, with $\epsilon^a \equiv a/|a|$. The commutation relations of the algebra have the form

$$\left[G^a_b, G^c_d\right] = \delta^c_b G^a_d - \delta^a_d G^c_b + \varepsilon^a\varepsilon^b \left(\delta^{-b}_d G^c_{-a} - \delta^c_{-a} G^{-b}_d\right). \tag{1}$$

In unitary representations $G^{a+}_b = G^b_a$. For convenience of the reader, the relation with the $O(5)$ Hermitian generators $L_{jk} = -\mathrm{i}(x_j\nabla_k - x_k\nabla_j)$, $j, k = 1, \ldots, 5$, is also given. Under the reduction $Sp(4) \supset SU(2) \times U(1)$, the $SU(2)$ generators are $J_0 = \left(G^1_1 - G^2_2\right)/2 = L_{34}$, $J_+ = G^1_2 = L_{45} + \mathrm{i}L_{53}$ and $J_- = G^2_1 = L_{45} - \mathrm{i}L_{53}$. The $U(1)$ generator is $H = \left(G^1_1 + G^2_2\right)/2 = L_{12}$. The remaining six generators define two irreducible vector operators with respect to the $SU(2)$ subgroup, one of which has the tensor components $U_1 = G^1_{-1}/\sqrt{2} = [L_{14}+L_{23}+\mathrm{i}(L_{24}+L_{31})]/\sqrt{2}$, $U_0 = G^1_{-2} = L_{52}+\mathrm{i}L_{15}$, $U_{-1} = G^2_{-2}/\sqrt{2} = [L_{14} - L_{23} + \mathrm{i}(L_{24} - L_{31})]/\sqrt{2}$, and the other, Hermitian conjugated to it, has the components $V_\kappa = (-1)^\kappa U^+_{-\kappa}$, $\kappa = 0, \pm 1$. The irreducible unitary representations can be labelled by $\langle q + p, q\rangle$, where $q + p \equiv 2J_m$ and $q \equiv 2\Lambda_m$ are the eigenvalues of the operators G^1_1 and G^2_2 in the highest weight state. The notation J_m, Λ_m was introduced by Hecht [1]. There are three available parameters for the identification of the basis states: the eigenvalues of the commuting Hermitian operators $\mathbf{J}^2$, J_0 and H, which will be denoted as $j(j + 1)$, m and τ. The necessary fourth parameter is missing; and this fact has been the essential problem in the construction of the non-canonical representations. In the following, the matrices of the $SU(2)$ generators will be assumed given in the standard form, the matrix of H will be diagonal and the vector operators U_κ and V_κ will be defined through their reduced matrix elements as in the expression (Wigner–Eckart)

$$\langle \alpha'\tau' j'm'|U_\kappa|\alpha\tau jm\rangle = (-1)^{j'-m'} \begin{pmatrix} j' & 1 & j \\ -m' & \kappa & m \end{pmatrix} \langle \alpha'\tau' j'\|\mathbf{U}\|\alpha\tau j\rangle. \tag{2}$$

In this definition the $\langle q + p, q\rangle$ dependence is implicit and the unknown label is denoted by α. These reduced matrix elements satisfy a set of equations which follow directly from the commutation relations of the algebra and are shown in the appendix.

In the following section the new method for the derivation of generic representations $\langle q + p, q\rangle$ is presented. It makes use of the reducible product of the multiplicity free representations $\langle p, 0\rangle \times \langle q, q\rangle$. In section 3 specific examples with two- and three-fold multiplicities are given.

2. The Bose realization

Boson realizations of Lie algebras have been applied extensively in the construction of arbitrary representations in suitable chosen Hilbert spaces of bosonic states. A comprehensive review of the different approaches, with applications to nuclear physics, was given by Klein and Marshalek [19]. In particular, a calculus of Bose operators was applied by Holman [20] in the construction of the canonical representations of $Sp(4)$, taking into consideration that $Sp(4)$ is itself a subgroup of $SU(4)$. In fact, the generators of $Sp(4)$ can be expressed as linear combinations of the Weyl generators E_{ij} of $SU(4)$:

$$H = (E_{11} - E_{22} + E_{33} - E_{44})/2 \tag{3}$$

$$J_0 = (E_{11} - E_{22} - E_{33} + E_{44})/2 \qquad J_+ = E_{13} - E_{42} \qquad J_- = E_{31} - E_{24} \tag{4}$$

$$U_0 = E_{14} + E_{32} \qquad U_1 = \sqrt{2}E_{12} \qquad U_{-1} = \sqrt{2}E_{34} \tag{5}$$

$$V_0 = E_{41} + E_{23} \qquad V_1 = -\sqrt{2}E_{43} \qquad V_{-1} = -\sqrt{2}E_{21}. \tag{6}$$

According to Holman, a Bose realization of Jordan–Schwinger type can be obtained by mapping the generators E_{ij} onto the following bilinear combinations of a set of boson creation a_i^b and destruction $\bar{a}_i^b$ operators ($i, j = 1, \ldots, 4$ and $b = 1, 2$):

$$E_{ij} = a_i^1\bar{a}_j^1 + a_i^2\bar{a}_j^2. \tag{7}$$

Basis states can be defined as certain polynomials in the four operators a_i^1 and the five double operators a_{14}, $a_{21} + a_{34}$, a_{23}, a_{13}, a_{24}, applied to the boson vacuum $|0\rangle$, where

$$a_{ij} = a_i^1 a_j^2 - a_j^1 a_i^2. \tag{8}$$

For further reference, the commutation relations obeyed by these operators are given here:

$$\left[E_{ij}, a_k^1\right] = \delta_{jk}a_i^1 \qquad \left[E_{ij}, \bar{a}_k^1\right] = -\delta_{ik}\bar{a}_j^1 \tag{9}$$

$$[E_{ij}, a_{kl}] = \delta_{jk}a_{il} + \delta_{jl}a_{ki} \qquad [E_{ij}, \bar{a}_{kl}] = -\delta_{ik}\bar{a}_{jl} - \delta_{il}\bar{a}_{kj} \tag{10}$$

$$[\bar{a}_{ij}, a_{kl}] = \delta_{ik}E_{lj} - \delta_{il}E_{kj} + \delta_{jl}E_{ki} - \delta_{jk}E_{li} + 2\delta_{ik}\delta_{jl} - 2\delta_{il}\delta_{jk}. \tag{11}$$

The explicit form of the basis in a generic canonical representation $\langle\sigma_1, \sigma_2\rangle$ was derived by Holman from the highest weight state $(a_{13})^{\sigma_2}\left(a_1^1\right)^{\sigma_1-\sigma_2}|0\rangle$ with the recursive action of the lowering operators introduced previously by Hecht [1].

In the case of the non-canonical reduction $Sp(4) \supset SU(2) \times U(1)$ a different approach must be developed, because of the problem of the missing label, which makes it impossible to obtain a suitable set of lowering operators: the multiplicity of states increases with the dimension of the representation. However, as is well known, the series of representations

$\langle p, 0\rangle$ and $\langle q, q\rangle (p, q = 1, 2, \ldots)$ are multiplicity free: the three labels j, m and τ completely specify the basis states and the matrices of generators are easily derived. Such representations have been obtained in the past by different methods. Here their Bose realization is derived and then used for the construction of generic representations $\langle q + p, q\rangle$.

2.1. The $\langle p, 0\rangle$ series

In a $\langle p, 0\rangle$ representation the state labels can take the following values: $j = p/2,\ p/2 - 1, \ldots, 0$ or $1/2$, for p even or odd; $m = -j, \ldots, j$ and $\tau = -j, \ldots, j$. These ranks of values suggest a relation of embedding of $\langle p, 0\rangle$ in the representations of the group of the symmetric top $SU(2) \times SU(2)$, which is defined by the direct product of two angular momentum algebras with the same value of total momentum. Indeed, we can identify the $U(1)$ operator H with the zero component of the second angular momentum operator K_0 and look for the representations of the wider group. The boson realization of $SU(2) \times SU(2)$ was given by Biedenharn and Louck [21]. Their result can be almost literally adapted to the realization of the $\langle p, 0\rangle$ representations of $sp(4)$. Indeed, the generators J_0, J_+ and J_- as well as $K_0 = H$ can be given as in equations (3) and (4) with $E_{ij} = a_i^1 \bar{a}_j^1$. The remaining (out of $sp(4)$ algebra) operators are $K_+ = \mathrm{i}\, a_3^1 \bar{a}_2^1 - \mathrm{i}\, a_1^1 \bar{a}_4^1$ and $K_- = (K_+)^+$. The orthonormal basis for the representations of $SU(2) \times SU(2)$ was defined in [21] as the following set of boson states, constructed only with the operators a_1^1, a_2^1, a_3^1 and a_4^1:

$$|\langle p, 0\rangle \tau j m\rangle = N_{p,j} (\det M)^{p/2-j} D^j_{m\tau}(M)|0\rangle. \tag{12}$$

In this definition M denotes the matrix

$$M = \begin{pmatrix} a_1^1 & \mathrm{i}a_4^1 \\ a_3^1 & -\mathrm{i}a_2^1 \end{pmatrix}.$$

The factor $D^j_{m\tau}(M)$ is the $SU(2)$ rotation matrix:

$$D^j_{m\tau}(M) = [(j+m)!(j-m)!(j+\tau)!(j-\tau)!]^{\frac{1}{2}} \times \sum_s \frac{\left(a_1^1\right)^{j+\tau-s} \left(a_3^1\right)^s \left(\mathrm{i}a_4^1\right)^{m-\tau+s} \left(-\mathrm{i}a_2^1\right)^{j-m-s}}{(j+\tau-s)!s!(m-\tau+s)!(j-m-s)!}.$$

Finally, the normalization coefficient is given by the equation

$$N_{p,j} = \left[(2j+1) \Big/ \left(\frac{p}{2} - j\right)! \left(\frac{p}{2} + j + 1\right)!\right]^{\frac{1}{2}}.$$

If operators U_κ of $sp(4)$ are defined as in (5) with $E_{ij} = a_i^1 \bar{a}_j^1$, then, as can be checked easily with the use of relations (9), their action on the basis vectors (12) leads to the well-known reduced matrix elements in $\langle p, 0\rangle$ (the dependence on parameter p is implicit):

$$\begin{aligned}
\langle \tau+1, j+1\|\mathbf{U}\|\tau j\rangle &= -\mathrm{i}\left[\frac{(p-2j)(p+2j+4)(j+\tau+1)(j+\tau+2)}{4(j+1)}\right]^{\frac{1}{2}} \\
\langle \tau+1, j\|\mathbf{U}\|\tau j\rangle &= \mathrm{i}\left[\frac{(p+2)^2(2j+1)(j+\tau+1)(j-\tau)}{4j(j+1)}\right]^{\frac{1}{2}} \\
\langle \tau+1, j-1\|\mathbf{U}\|\tau j\rangle &= -\mathrm{i}\left[\frac{(p-2j+2)(p+2j+2)(j-\tau-1)(j-\tau)}{4j}\right]^{\frac{1}{2}}.
\end{aligned} \tag{13}$$

The reduced matrix elements of the Hermitian conjugated vector operator $\mathbf{V} = \mathbf{U}^+$ follow from the relation

$$\langle \tau' j'\|\mathbf{V}\|\tau j\rangle = (-1)^{j-j'} \langle \tau j\|\mathbf{U}\|\tau' j'\rangle^*. \tag{14}$$

In summary, the basis (12) and the operators (3)–(6) with $E_{ij} = a_i^1 \bar{a}_j^1$ give a realization of the series of representations $\langle p, 0\rangle$ of $sp(4)$, with p any positive integer.

2.2. The $\langle q, q\rangle$ series

In a representation $\langle q, q\rangle$ the possible values of the parameters are $j = q, q-1, \ldots, 0$; $m = -j, \ldots, j$ and $\tau = q-j, q-j-2, \ldots, -(q-j)$. So, for each value of j, the parameter τ takes only even values or only odd values, depending on q. Since operators U_κ rise and operators V_κ lower the eigenvalues τ of the operator H by unity, changing their parity, the matrix elements of U_κ and V_κ with $j \to j$ transitions are equal to zero. The non-zero reduced matrix elements can be obtained by solving equations (A.10)–(A.13) given in the appendix:

$$\begin{aligned}\langle \tau+1, j+1\|\mathbf{U}\|\tau j\rangle &= [(j+1)(q+j+\tau+3)(q-j-\tau)]^{\frac{1}{2}} \\ \langle \tau+1, j-1\|\mathbf{U}\|\tau j\rangle &= [j(q+j-\tau+1)(q-j+\tau+2)]^{\frac{1}{2}}\end{aligned} . \tag{15}$$

The reduced matrix elements of operator $\mathbf{V}$ follow immediately from (14).

The Bose realization of the representations $\langle q, q\rangle$ can now be derived with the recurrent action of the operators V_1, V_{-1} and J_- onto the highest weight state

$$|\tau = q, j = 0, m = 0\rangle = [q!(q+1)!]^{-\frac{1}{2}}(a_{13})^q|0\rangle. \tag{16}$$

The values of parameters τ, j and m are obtained applying to the right-hand side H, $\mathbf{J}^2$ and J_0 defined as in (3) and (4), with $E_{ij} = a_i^1\bar{a}_j^1 + a_i^2\bar{a}_j^2$. This and subsequent computations are based on the use of the commutation relations (10).

The explicit form of matrix elements of operator V_1 can be used in the derivation of the expression

$$|\tau = q-j, j, m = j\rangle = (-1)^j \left[\frac{(q-j)!}{q!j!2^j}\right]^{\frac{1}{2}} (V_1)^j|\tau = q, j = 0, m = 0\rangle. \tag{17}$$

Computing the right-hand side with the bosonic version of the operators and vectors, this equation takes the form

$$|\tau = q-j, j, m = j\rangle = \frac{(a_{13})^{q-j}(a_{14})^j}{\sqrt{(q+1)!(q-j)!j!}}|0\rangle. \tag{18}$$

The following relation can be obtained in complete analogy (here $k \equiv (q-j)/2$):

$$\begin{aligned}|\tau, j, m = j\rangle &= \left[\frac{(q+j+\tau+2)!(q+j-\tau+1)!\left(k+\frac{\tau}{2}\right)!\left(k-\frac{\tau}{2}\right)!(2j+1)!q!}{\left(q-k+\frac{\tau}{2}+1\right)!\left(k+j-\frac{\tau}{2}\right)!(2q+1)!2^{k-\frac{\tau}{2}+1}}\right]^{\frac{1}{2}}(-1)^{k-\frac{\tau}{2}} \\ &\times \sum_{z=0}^{\min(k-\frac{\tau}{2},k+\frac{\tau}{2})} \frac{2^{\frac{z}{2}}(j+z)!(V_{-1})^{k-\frac{\tau}{2}-z}(J_-)^{2z}}{z!\left(k-\frac{\tau}{2}-z\right)!(2j+1+2z)!\left[\left(j+k-\frac{\tau}{2}+z\right)!\left(k+\frac{\tau}{2}-z\right)!\right]^{\frac{1}{2}}} \\ &\times \left|q-\left(j+k-\frac{\tau}{2}+z\right), j+k-\frac{\tau}{2}+z, m = j+k-\frac{\tau}{2}+z\right\rangle.\end{aligned} \tag{19}$$

Once again, the use of the Bose operators and vectors leads to the following explicit expression, which represents a generic basis vector for the $\langle q, q\rangle$ representation:

$$\begin{aligned}|\langle q, q\rangle\tau \equiv 2n, j, m\rangle &= N_{qnjm}\sum_{x,y,z}\frac{(-1)^z(j+z)!(j-m+2z)!(a_{13})^{2n+y}(a_{24})^y}{z!(2j+1+2z)!(2n+y)!y!(k-n-z-y)!} \\ &\times \frac{(a_{14})^{k-n+m-z-y+x}(a_{21}+a_{34})^{j-m+2z-2x}(a_{23})^{k-n-z-y+x}}{(k-n+m-z-y+x)!(j-m+2z-2x)!x!}|0\rangle.\end{aligned} \tag{20}$$

The parameter $n \equiv \tau/2$ was introduced for convenience; it takes the values $k, k-1, \ldots, -k$, with $k = (q-j)/2$. The sums are over all integer values allowed by the conditions: $0 \leqslant z \leqslant \min(k-n, k+n)$, $\min(-2n, 0) \leqslant y \leqslant k-n-z$, $\max(0, n-k-m+z+y) \leqslant x \leqslant z+(j-m)/2$. The normalization factor in (20) is

$$N_{qnjm} = \left[\frac{(j+m)!(k+n)!(k-n)!(q+j+2n+2)!(q+j-2n+1)!(2j+1)}{(j-m)!(q-k+n+1)!(k+j-n)!(2q+2)!}\right]^{\frac{1}{2}}.$$

The basis (20) and the operators (3)–(6), with $E_{ij} = a_i^1\bar{a}_j^1 + a_i^2\bar{a}_j^2$, give the Bose realization of the series of representations $\langle q, q\rangle$, with q any positive integer.

2.3. The general case

In this subsection the proposed method of derivation of a generic representation $\langle q+p, q\rangle$ is presented. The vectors of the basis will be obtained as linear combinations of the basis vectors of the reducible product of representations $\langle p, 0\rangle \times \langle q, q\rangle$. From now on, the explicit labelling of specific representations cannot be avoided without confusion. It will be given, when necessary, in the following short form: $p0$ for $\langle p, 0\rangle$ and qq for $\langle q, q\rangle$.

The basis of $\langle p, 0\rangle \times \langle q, q\rangle$ will be defined in standard form:

$$|pqn_1n_2j_1j_2nJM\rangle = \sum_{m_1m_2} \langle j_1j_2m_1m_2 \mid JM\rangle |p0, n_1j_1m_1\rangle |qq, n_2j_2m_2\rangle. \tag{21}$$

The parameters $n_1 \equiv \tau_1$ and $n_2 \equiv \tau_2/2$ were introduced for convenience. The eigenvalue of the operator H is denoted by $n = n_1 + 2n_2$. The parameter J takes the values $j_1 + j_2$, $j_1 + j_2 - 1, \ldots, |j_1 - j_2|$. As usual, $M \equiv m_1 + m_2$.

The vectors on the right-hand side of (21) depend on two different (commuting) sets of operators: $a_1^1(1)$, $a_2^1(1)$, $a_3^1(1)$, $a_4^1(1)$ for the $\langle p, 0\rangle$ representation and $a_{13}(2)$, $a_{24}(2)$, $a_{14}(2)$, $a_{21}(2) + a_{34}(2)$, $a_{23}(2)$, for the $\langle q, q\rangle$ representation. They are defined as in (12) and (20). The generators of the algebra *sp*(4) still have the form (3)–(6), but now the operators E_{ij} in equation (7) are a sum of two commuting terms, $E_{ij} = E_{ij}(1) + E_{ij}(2)$, the first operating on $\langle p, 0\rangle$ and the second on $\langle q, q\rangle$. In the Bose realization $E_{ij}(1) = a_i^1(1)\bar{a}_j^1(1)$ and $E_{ij}(2) = a_i^1(2)\bar{a}_j^1(2) + a_i^2(2)\bar{a}_j^2(2)$.

The matrix elements of operators $\mathbf{U}$ and $\mathbf{V}$ in $\langle p, 0\rangle \times \langle q, q\rangle$ can be computed with the standard expressions of the *su*(2) tensor algebra. For example, the operator $\mathbf{U} = \mathbf{U}(1) \times \mathbf{I}(2) + \mathbf{I}(1) \times \mathbf{U}(2)$ has the following reduced matrix elements:

$$\begin{aligned}\langle pqn_1'n_2'j_1'j_2'J'\|\mathbf{U}\|pqn_1n_2j_1j_2J\rangle &= (-1)^{j_1'+1}\sqrt{(2J'+1)(2J+1)}\\ &\times\left[\delta_{n_2'n_2}\delta_{j_2'j_2}(-1)^{J+j_2'}\begin{pmatrix} j_2' & J & j_1\\ 1 & j_1' & J'\end{pmatrix}\langle p0, n_1'j_1'\|\mathbf{U}\|p0, n_1j_1\rangle\right.\\ &\left.+\,\delta_{n_1'n_1}\delta_{j_1'j_1}(-1)^{J'+j_2}\begin{pmatrix} j_1' & J & j_2\\ 1 & j_2' & J'\end{pmatrix}\langle qq, n_2'j_2'\|\mathbf{U}\|qq, n_2j_2\rangle\right].\end{aligned} \tag{22}$$

The matrix elements on the right-hand side are given by equations (13) and (15).

The generic irreducible representation $\langle q+p, q\rangle$ is contained in the reducible product of representations $\langle p, 0\rangle \times \langle q, q\rangle$. Indeed, the highest weight vector in $\langle q+p, q\rangle$ is the product of the highest weight vectors of the factors $\langle p, 0\rangle$ and $\langle q, q\rangle$. All other vectors in $\langle q+p, q\rangle$, and only these, can be reached by the action of the *sp*(4) generators and their powers onto the highest weight, which is represented by the following not-degenerated vector:

$$\begin{aligned}\left|\langle q+p, q\rangle n = q+\frac{p}{2}, J = \frac{p}{2}, M = \frac{p}{2}\right\rangle &= \frac{\left[a_1^1(1)\right]^p[a_{13}(2)]^q}{\sqrt{p!q!(q+1)!}}|0\rangle\\ &= \left|pq, n_1 = \frac{p}{2}, n_2 = \frac{q}{2}, j_1 = \frac{p}{2}, j_2 = 0, n = q+\frac{p}{2}, J = M = \frac{p}{2}\right\rangle.\end{aligned} \tag{23}$$

The generic basis vectors of $\langle q+p, q\rangle$ will be defined as the linear combination

$$|\langle p+q, q\rangle \alpha nJM\rangle = \sum_{\{n_1 n_2 j_1 j_2\}} |pqn_1 n_2 j_1 j_2 nJM\rangle A_{\alpha nJ}(n_1 n_2 j_1 j_2). \tag{24}$$

The parameter α is introduced here to label the multiplicity of these vectors. The sum in (24) is taken over all the sets of parameters $\{n_1 n_2 j_1 j_2\}$ that lead to the same values nJ. In the cases with only one such set, the coefficient A is irrelevant and can be chosen equal to unity.

The coefficients A in the expansion (24) will be determined following a method similar to the one earlier applied by Sharp and Pieper in the construction of the canonical representations on polynomial bases [2]. They proved that the polynomials corresponding to the $\langle 1, 0\rangle$ term in the reduction $\langle 1, 0\rangle \times \langle 1, 1\rangle = \langle 2, 1\rangle + \langle 1, 0\rangle$ must be orthogonal to every state of the $\langle p+q, q\rangle$ term in the reduction of the $\langle p, 0\rangle \times \langle q, q\rangle$ representation.

In the present Bose realization of non-canonical representations, the basis of such a $\langle 1, 0\rangle$ term can be found in the form $\Delta_i^+|0\rangle/\sqrt{20}$, $i = 1, \ldots, 4$, where

$$\begin{aligned}
\Delta_1^+ &= \left\{a_1^1(1)[a_{21}(2) + a_{34}(2)] - 2a_3^1(1)a_{14}(2) + 2a_4^1(1)a_{13}(2)\right\} \\
\Delta_2^+ &= -\left\{a_3^1(1)[a_{21}(2) + a_{34}(2)] - 2a_1^1(1)a_{23}(2) + 2a_2^1(1)a_{13}(2)\right\} \\
\Delta_3^+ &= -\mathrm{i}\left\{a_4^1(1)[a_{21}(2) + a_{34}(2)] + 2a_2^1(1)a_{14}(2) - 2a_1^1(1)a_{24}(2)\right\} \\
\Delta_4^+ &= -\mathrm{i}\left\{a_2^1(1)[a_{21}(2) + a_{34}(2)] + 2a_4^1(1)a_{23}(2) - 2a_3^1(1)a_{24}(2)\right\}.
\end{aligned} \tag{25}$$

The condition of orthogonality $\langle 0|\Delta_i|\langle p+q, q\rangle \alpha nJM\rangle = 0$ means that the Hermitian conjugated operators Δ_i should annihilate every state of the $\langle p+q, q\rangle$ term in the reduction of $\langle p, 0\rangle \times \langle q, q\rangle$. This will be the case if they annihilate at least one such state. That happens, for example, with the highest weight state (23). In fact, the use of commutation relations (11) gives ($i = 1, \ldots, 4$)

$$\Delta_i \left| \langle q+p, q\rangle n = q + \frac{p}{2}, J = \frac{p}{2}, M = \frac{p}{2} \right\rangle = \Delta_i \frac{\left[a_1^1(1)\right]^p [a_{13}(2)]^q}{\sqrt{p!q!(q+1)!}} |0\rangle = 0. \tag{26}$$

Now, let G be any of the generators of the algebra and denote as $|HW\rangle$ the highest weight vector (23). Since $\Delta_i|HW\rangle = 0$, then

$$\Delta_i G|HW\rangle = [\Delta_i, G]|HW\rangle + G\Delta_i|HW\rangle = 0 \tag{27}$$

because the commutator $[\Delta_i, G]$ is a linear combination of the operators Δ_i themselves. By induction, the operators Δ_i annihilate every state of the representation $\langle q+p, p\rangle$ as all of them can be reached by the successive action of the generators on $|HW\rangle$.

Let us apply this statement to the basis vectors (24):

$$\Delta_i|\langle p+q, q\rangle \alpha nJM\rangle = \sum_{\{n_1 n_2 j_1 j_2\}} \Delta_i |pqn_1 n_2 j_1 j_2 nJM\rangle A_{\alpha nJ}(n_1 n_2 j_1 j_2) = 0. \tag{28}$$

This condition originates a system of linear homogeneous algebraic equations for the unknown coefficients A, which could be explicitly written if the matrix elements of Δ_i in $\langle p, 0\rangle \times \langle q, q\rangle$ were known. The use of commutation relations (9) and (10) allows us to prove that the operators Δ_i represent the components of two irreducible tensor operators $\mathbf{B}^{(1/2)}$ and $\mathbf{C}^{(1/2)}$ with respect to the $SU(2)$ subgroup: $B_{1/2}^{(1/2)} = \Delta_2/\sqrt{2}$, $B_{-1/2}^{(1/2)} = -\Delta_1/\sqrt{2}$ and $C_{1/2}^{(1/2)} = \mathrm{i}\Delta_4/\sqrt{2}$, $C_{-1/2}^{(1/2)} = -\mathrm{i}\Delta_3/\sqrt{2}$. Then, equation (28) can be simplified applying the Wigner–Eckart theorem:

$$\sum_{\{n_1 n_2 j_1 j_2\}} \left\langle p'q'n_1'n_2'j_1'j_2'n'J' \left\| \mathbf{B}^{(1/2)} \right\| pqn_1 n_2 j_1 j_2 nJ \right\rangle A_{\alpha nJ}(n_1 n_2 j_1 j_2) = 0 \tag{29}$$

$$\sum_{\{n_1 n_2 j_1 j_2\}} \left\langle p'q'n_1'n_2'j_1'j_2'n'J' \left\| \mathbf{C}^{(1/2)} \right\| pqn_1 n_2 j_1 j_2 nJ \right\rangle A_{\alpha nJ}(n_1 n_2 j_1 j_2) = 0. \tag{30}$$

The operators $\mathbf{B}^{(1/2)}$ and $\mathbf{C}^{(1/2)}$ themselves can be expressed as $SU(2)$ tensor products:

$$\mathbf{B}^{(1/2)} = \sqrt{3}\left[\mathbf{Q}^{(1/2)} \times \mathbf{S}^{(1)}\right]^{(1/2)} - \sqrt{2}\mathbf{R}^{(1/2)}\bar{a}_{13}(2) \tag{31}$$

$$\mathbf{C}^{(1/2)} = \sqrt{3}\left[\mathbf{R}^{(1/2)} \times \mathbf{S}^{(1)}\right]^{(1/2)} + \sqrt{2}\mathbf{Q}^{(1/2)}\bar{a}_{24}(2). \tag{32}$$

In these products the following factors operate on $\langle p, 0\rangle$:

$$Q^{(1/2)}_{1/2} = \bar{a}^1_3(1) \qquad Q^{(1/2)}_{-1/2} = -\bar{a}^1_1(1) \qquad R^{(1/2)}_{1/2} = \bar{a}^1_2(1) \qquad R^{(1/2)}_{-1/2} = \bar{a}^1_4(1) \tag{33}$$

and the next ones operate on $\langle q, q\rangle$:

$$S^{(1)}_1 = \bar{a}_{23}(2) \qquad S^{(1)}_0 = [\bar{a}_{12}(2) + \bar{a}_{43}(2)]/\sqrt{2} \qquad S^{(1)}_{-1} = \bar{a}_{14}(2). \tag{34}$$

The remaining factors $\bar{a}_{13}(2)$ and $\bar{a}_{24}(2)$ behave as scalars with respect to the subgroup $SU(2)$ and also operate on $\langle q, q\rangle$. The operators $\bar{a}_{ij}$ are Hermitian conjugated to the a_{ij} defined by equation (8).

The reduced matrix elements in (29) and (30) can be calculated using the composition relations for the product of two $SU(2)$ irreducible tensor operators. For example,

$$\begin{aligned}\left\langle p'q'n'_1n'_2j'_1j'_2n'J' \left\|\mathbf{B}^{(1/2)}\right\| pqn_1n_2j_1j_2nJ\right\rangle &= \sqrt{2(2J'+1)(2J+1)} \\ &\times\left[\sqrt{3}\begin{Bmatrix} j_1 & j_2 & J \\ \frac{1}{2} & 1 & \frac{1}{2} \\ j'_1 & j'_2 & J'\end{Bmatrix}\left\langle p'0, n'_1j'_1 \left\|\mathbf{Q}^{(1/2)}\right\| p0, n_1j_1\right\rangle\left\langle q'q', n'_2j'_2 \left\|\mathbf{S}^{(1)}\right\| qq, n_2j_2\right\rangle\right. \\ &\left. -\sqrt{2}\begin{Bmatrix} j_1 & j_2 & J \\ \frac{1}{2} & 0 & \frac{1}{2} \\ j'_1 & j'_2 & J'\end{Bmatrix}\left\langle p'0, n'_1j'_1 \left\|\mathbf{R}^{(1/2)}\right\| p0, n_1j_1\right\rangle\left\langle q'q', n'_2j'_2\|\bar{a}_{13}\|qq, n_2j_2\right\rangle\right].\end{aligned} \tag{35}$$

A similar expression can be given for the tensor $\mathbf{C}^{(1/2)}$. Hence, it only remains to calculate the reduced matrix elements of the different factors in expressions (31) and (32). This can be done using the Bose realization of the previous subsections. The non-zero matrix elements are (here $p' = p - 1$, $q' = q - 1$)

$$\left\langle p'0, n - \frac{1}{2}, j + \frac{1}{2} \left\|\mathbf{Q}^{(1/2)}\right\| p0, nj\right\rangle = \left[\left(\frac{p}{2} - j\right)(j - n + 1)\right]^{\frac{1}{2}} \tag{36}$$

$$\left\langle p'0, n - \frac{1}{2}, j - \frac{1}{2} \left\|\mathbf{Q}^{(1/2)}\right\| p0, nj\right\rangle = \left[\left(\frac{p}{2} + j + 1\right)(j + n)\right]^{\frac{1}{2}} \tag{37}$$

$$\left\langle p'0, n + \frac{1}{2}, j + \frac{1}{2} \left\|\mathbf{R}^{(1/2)}\right\| p0, nj\right\rangle = \mathrm{i}\left[\left(\frac{p}{2} - j\right)(j + n + 1)\right]^{\frac{1}{2}} \tag{38}$$

$$\left\langle p'0, n + \frac{1}{2}, j - \frac{1}{2} \left\|\mathbf{R}^{(1/2)}\right\| p0, nj\right\rangle = -\mathrm{i}\left[\left(\frac{p}{2} + j + 1\right)(j - n)\right]^{\frac{1}{2}} \tag{39}$$

$$\left\langle q'q', n, j + 1 \left\|\mathbf{S}^{(1)}\right\| qq, nj\right\rangle = \left[\frac{(j+1)(q+1)(q-j+2n)(q-j-2n)}{2q+1}\right]^{\frac{1}{2}} \tag{40}$$

$$\left\langle q'q', n, j - 1 \left\|\mathbf{S}^{(1)}\right\| qq, nj\right\rangle = \left[\frac{j\,(q+1)(q+j+2n+1)(q+j-2n+1)}{2q+1}\right]^{\frac{1}{2}} \tag{41}$$

$$\left\langle q'q', n - \frac{1}{2}, j \left\|\bar{a}_{13}\right\| qq, nj\right\rangle = \left[\frac{(2j+1)(q+1)(q+j+2n+1)(q-j+2n)}{2(2q+1)}\right]^{\frac{1}{2}} \tag{42}$$

$$\left\langle q'q', n + \frac{1}{2}, j \left\|\bar{a}_{24}\right\| qq, nj\right\rangle = \left[\frac{(2j+1)(q+1)(q+j-2n+1)(q-j-2n)}{2(2q+1)}\right]^{\frac{1}{2}}. \tag{43}$$

With these expressions the coefficients in equations (29) and (30) become explicitly defined. It must be remarked that there $p' = p - 1$, $q' = q - 1$ and $J' = J \pm 1/2$. In general, there are up to four equations for each possible set of parameters n_1', n_2', j_1', j_2', n', J' defined in the basis $\langle p-1, 0\rangle \times \langle q-1, q-1\rangle$.

The condition of orthonormality of the basis (24) is an additional requirement given by the equation

$$\sum_{\{n_1 n_2 j_1 j_2\}} A^*_{\alpha' n J}(n_1 n_2 j_1 j_2) A_{\alpha n J}(n_1 n_2 j_1 j_2) = \delta_{\alpha' \alpha} \tag{44}$$

which leads to the final solutions $A_{\alpha n J}(n_1 n_2 j_1 j_2)$: (i) when, for specific values of n and J, the unknowns A_{nJ} do not appear in equations (29) and (30), and necessarily the set $\{n_1 n_2 j_1 j_2\}$ corresponding to those values is unique, condition (44) determines those unknowns up to a phase factor. They can be chosen equal to unity. (ii) When the range of the system of equations (29) and (30) is by unity less than the number of different sets $\{n_1 n_2 j_1 j_2\}$ related to a given pair n, J, then condition (44) gives a unique solution for the unknowns $A_{nJ}(n_1 n_2 j_1 j_2)$ up to a global phase factor. (iii) When the range of that system is by two or more unities less than the number of different sets $\{n_1 n_2 j_1 j_2\}$, then the solution for the unknowns $A_{\alpha n J}(n_1 n_2 j_1 j_2)$ is degenerate. The parameter α was introduced to enumerate such degeneracies. Within each degenerated multiplet the freedom to perform unitary transformations preserving (44) remains.

Once the unknowns A were found from equations (29) and (30), the reduced matrix elements of the operator $\mathbf{U}$ can be computed using (22) in the expression

$$\begin{aligned}\langle\langle p+q, q\rangle\alpha' n' J' \| \mathbf{U} \| \langle p+q, q\rangle \alpha n J\rangle &= \sum\sum A^*_{\alpha' n' J'}(n_1' n_2' j_1' j_2') \\ &\times \langle pq n_1' n_2' j_1' j_2' J' \| \mathbf{U} \| pq n_1 n_2 j_1 j_2 J\rangle A_{\alpha n J}(n_1 n_2 j_1 j_2).\end{aligned} \tag{45}$$

Here the sums are defined over the sets $\{n_1' n_2' j_1' j_2'\}$ and $\{n_1 n_2 j_1 j_2\}$ as above.

The reduced matrix elements of the operator $\mathbf{V}$ follow immediately from $\langle \alpha' n' J' \| \mathbf{V} \| \alpha n J\rangle = (-1)^{J-J'} \langle \alpha n J \| \mathbf{U} \| \alpha' n' J'\rangle^*$.

In the next section explicit results of these computations are presented, including the series of representations $\langle q+4, q\rangle$, which has not been derived by other methods.

3. Examples

As was mentioned above, representations with multiplicities of two or less were derived by other authors in the past using different methods. We present our derivation for comparison and completeness. The representations with multiplicities greater than two were difficult to deal with by other methods. The series of representations $\langle q+4, q\rangle$, with multiplicities three or less, a result that has not been reported in the past, is presented here.

For the convenience of the reader, let us recall that the labelling of the equivalent representations of $O(5)$ is given by (ω_1, ω_2), where now $\omega_1 = q + \frac{p}{2}$ and $\omega_2 = \frac{p}{2}$. In the works of Hecht and co-authors, the parameter J is the total isotopic spin and is represented as T, the parameter n is related to the number of particle eigenvalues and is represented as H_1. Our reduced matrix elements of the operator $\mathbf{U}$ are related to their $R(5)$ Wigner coefficients by the equivalence

$$\begin{aligned}\langle\langle q+p, q\rangle \alpha' H_1' T' \| \mathbf{U} \| \langle q+p, q\rangle \alpha H_1 T\rangle &\sim \{2(2T+1)[\omega_1(\omega_1+3) + \omega_2(\omega_2+1)]\}^{\frac{1}{2}} \\ &\times \langle(\omega_1, \omega_2) k H_1 T; (11) 11 \| (\omega_1, \omega_2) k' H_1' T'\rangle.\end{aligned}$$

As can be verified, the different methods lead to the same matrix elements only in the cases of multiplicity one up to trivial phase factors. In the case of greater multiplicities the

Table 1. $\langle\langle q+1, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+1, q\rangle\alpha, n, J\rangle$.

α'	α	k	$q+J+n=$ even	$q+J+n=$ odd
1	1	1	$\left[\frac{(2J+1)(2J+3)(q-J-n)(q+J+n+3)}{4(J+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(2J+1)(2J+3)(q-J-n+1)(q+J+n+4)}{4(J+1)}\right]^{\frac{1}{2}}$
1	1	-1	$\left[\frac{(2J-1)(2J+1)(q-J+n+3)(q+J-n+2)}{4J}\right]^{\frac{1}{2}}$	$\left[\frac{(2J-1)(2J+1)(q-J+n+2)(q+J-n+1)}{4J}\right]^{\frac{1}{2}}$
1	1	0	$\mathrm{i}\left[\frac{(2J+1)(q+J-n+2)(q+J+n+3)}{4J(J+1)}\right]^{\frac{1}{2}}$	$\mathrm{i}\left[\frac{(2J+1)(q-J-n+1)(q-J+n+2)}{4J(J+1)}\right]^{\frac{1}{2}}$

Table 2. $\langle\langle q+2, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+2, q\rangle\alpha, n, J\rangle^{\mathrm{a}}$.

α'	α	k	$q+J+n=$ even
1	1	1	$\left[\frac{(J+1)(q-J-n+2)(q+J+n+5)[(q+1)(q+2)(q+3)-(q+3)n(n+1)-(q+2)J(J+2)]^2}{F(q,n+1,J+1)F(q,n,J)}\right]^{\frac{1}{2}}$
1	2	1	$\left[\frac{J(q+2)(q+3)(q-J+n+2)(q+j-n+3)(q+J+n+3)(q+J+n+5)}{F(q,n+1,J+1)F(q,n,J)}\right]^{\frac{1}{2}}$
2	1	1	$\left[\frac{(J+2)(q+2)(q+3)(q-J-n)(q-J-n+2)(q-J+n+2)(q+J-n+3)}{F(q,n+1,J+1)F(q,n,J)}\right]^{\frac{1}{2}}$
2	2	1	$\left[\frac{J(J+2)(q-J-n)(q+J+n+3)[(q+2)(q+4)^2-(q+3)n(n+1)-(q+2)J(J+2)]^2}{(J+1)F(q,n+1,J+1)F(q,n,J)}\right]^{\frac{1}{2}}$
1	1	-1	$\left[\frac{J(q-J+n+4)(q+J-n+3)[(q+2)^3-(q+3)n(n+1)-(q+2)J^2]^2}{F(q,n+1,J-1)F(q,n,J)}\right]^{\frac{1}{2}}$
1	2	-1	$-\left[\frac{(J+1)(q+2)(q+3)(q-J-n+2)(q-J+n+2)(q-J+n+4)(q+J+n+3)}{F(q,n+1,J-1)F(q,n,J)}\right]^{\frac{1}{2}}$
2	1	-1	$-\left[\frac{(J-1)(q+2)(q+3)(q-J-n+2)(q+J-n+1)(q+J-n+3)(q+J+n+3)}{F(q,n+1,J-1)F(q,n,J)}\right]^{\frac{1}{2}}$
2	2	-1	$\left[\frac{(J-1)(J+1)(q-J+n+2)(q+J-n+1)[(q+2)(q+3)^2-(q+3)n(n+1)-(q+2)J^2]^2}{JF(q,n+1,J-1)F(q,n,J)}\right]^{\frac{1}{2}}$
1	1	0	$-\left[\frac{(2J+1)(q+2)(q-J-n+2)(q+J-n+3)}{F(q,n,J)}\right]^{\frac{1}{2}}$
1	2	0	$\left[\frac{(2J+1)(q+3)(q-J+n+2)(q+J+n+3)(q-n+2)^2}{J(J+1)F(q,n,J)}\right]^{\frac{1}{2}}$

[a] $F(q, n, J) = (q+2)^2(q+3) - (q+3)n^2 - (q+2)J(J+1)$.

matrices are different because of the different labelling scheme. They should be related by unitary transformations.

The expressions given below were checked with the equations listed in the appendix, which follow directly from the commutation relations of the algebra. They are given for states which have, in general, multiplicities greater than 1. For states with lesser multiplicities (perimeter or next to perimeter states) the allowed values of parameter α are explicitly stated. These values follow from the unique assumption that $\alpha = 1$ in the corner states with $J = \frac{p}{2}$ and $n = -\left(q + \frac{p}{2}\right)$, since all the other states can be reached from these corner states by the action of the operator $\mathbf{U}$ (and the trivial action of the operator $\mathbf{J}$). Let us recall once again that the parameter q can take any positive integer value.

Series $\langle q+1, q\rangle$. The non-zero reduced matrix elements of the operator $\mathbf{U}$ are presented in table 1. This is a simple case with multiplicities equal to 1. The allowed values of the parameters are $J = \frac{1}{2}, \frac{3}{2}, \ldots, q + \frac{1}{2}$ and $n = (q-J+1), (q-J+1)-1, \ldots, -(q-J+1)$.

Series $\langle q+2, q\rangle$. In this case the values of the parameters are: $J = 0, 1, \ldots, q+1$ and $n = (q-J+2), (q-J+2)-1, \ldots, -(q-J+2)$ for $J \neq 0$, $n = q, q-2, \ldots, -q$ for $J = 0$. The reduced matrix elements of the operator $\mathbf{U}$ applied to basis vectors with parameters satisfying the condition $q+J+n =$ even, are presented in table 2. Those vectors

Table 3. $\langle\langle q+2, q\rangle\, \alpha', n+1, J+k \,\|\mathbf{U}\|\, \langle q+2, q\rangle\, \alpha, n, J\rangle$[a].

α'	α	k	$q+J+n=$ odd
1	1	1	$\left[\frac{J(J+2)(q-J-n+1)(q+J+n+4)}{J+1}\right]^{\frac{1}{2}}$
1	1	−1	$\left[\frac{(J-1)(J+1)(q-J+n+3)(q+J-n+2)}{J}\right]^{\frac{1}{2}}$
1	1	0	$-\left[\frac{(2J+1)(q+2)(q-J+n+3)(q+J+n+4)}{F(q,n+1,J)}\right]^{\frac{1}{2}}$
2	1	0	$-\left[\frac{(2J+1)(q+3)(q-J-n+1)(q+J-n+2)(q+n+3)^2}{J(J+1)F(q,n+1,J)}\right]^{\frac{1}{2}}$

[a] $F(q, n, J)$ as in table 2.

Table 4. $\langle\langle q+3, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+3, q\rangle\alpha, n, J\rangle$[a].

α'	α	k	$q+J+n=$ even
1	1	1	$\left[\frac{(2J+1)(2J+5)(q-J-n)(q+J+n+5)G(q,n,J)}{4(J+1)G(q,n+1,J+1)}\right]^{\frac{1}{2}}$
1	2	1	0
2	1	1	$-\left[\frac{12(q+2)(q+4)(q-J+n+3)(q+J-n+4)(q+3j+n+7)^2}{(J+1)G(q,n+1,J+1)G(q,n,J)}\right]^{\frac{1}{2}}$
2	2	1	$\left[\frac{(2J-1)(2J+3)(q-J-n+2)(q+J+n+3)G(q,n+1,J+1)}{4(J+1)G(q,n,J)}\right]^{\frac{1}{2}}$
1	1	−1	$\left[\frac{(2J-1)(2J+3)(q-J+n+5)(q+J-n+4)G(q,n+1,J-1)}{4JG(q,n,J)}\right]^{\frac{1}{2}}$
1	2	−1	$-\left[\frac{12(q+2)(q+4)(q-J-n+2)(q-J+n+3)(q-J+n+5)(q+J+n+3)}{JG(q,n+1,J-1)G(q,n,J)}\right]^{\frac{1}{2}}$
2	1	−1	0
2	2	−1	$\left[\frac{(2J-3)(2J+1)(q-J+n+3)(q+J-n+2)G(q,n,J)}{4JG(q,n+1,J-1)}\right]^{\frac{1}{2}}$
1	1	0	$-\mathrm{i}\left[\frac{9(2J+1)(q-J-n+2)(q-J+n+3)(q+J-n+4)(q+J+n+5)(2q+2j+7)^2}{4J(J+1)G(q,-n-1,J)G(q,n,J)}\right]^{\frac{1}{2}}$
1	2	0	$\mathrm{i}\left[\frac{3(2J-1)(2J+1)(2J+3)(q+2)(q+4)(q+J+n+3)(q+J+n+5)(q+J-n+4)^2}{J(J+1)G(q,-n-1,J)G(q,n,J)}\right]^{\frac{1}{2}}$
2	1	0	$\mathrm{i}\left[\frac{3(2J-1)(2J+1)(2J+3)(q+2)(q+4)(q+J-n+2)(q+J-n+4)(q+J+n+5)^2}{J(J+1)G(q,-n-1,J)G(q,n,J)}\right]^{\frac{1}{2}}$
2	2	0	$\mathrm{i}\left[\frac{(2J+1)(q-J-n+2)(q-J+n+3)(q+J-n+2)(q+J+n+3)[14q+55+2(4q+17)J]^2}{4J(J+1)G(q,-n-1,J)G(q,n,J)}\right]^{\frac{1}{2}}$

[a] $G(q, n, J) = (q+2)(4q+15) + (4q+17)n + [4q^2 + 30q + 53 + 2(2q+7)(J+n)]J$.

have in general multiplicities equal to two, excluding the states with $J = 0$ and the states with $J \pm n = q+2$, which are the perimeter states. In the perimeter states $\alpha = 1$. The basis vectors with $q + J + n =$ odd are not degenerate and the corresponding matrix elements are shown in table 3.

Series $\langle q+3, q\rangle$. Here the parameters take the values $J = \frac{1}{2}, \frac{3}{2}, \ldots, q + \frac{3}{2}$ and $n = (q - J + 3), (q - J + 3) - 1, \ldots, -(q - J + 3)$ for $J \neq \frac{1}{2}$, $n = \left(q + \frac{1}{2}\right), \left(q + \frac{1}{2}\right) - 1, \ldots, -\left(q + \frac{1}{2}\right)$ for $J = \frac{1}{2}$. The reduced matrix elements are presented in tables 4 and 5, where $q + J + n$ is an even and odd integer respectively. In general, the multiplicity of the basis vectors is two, with the exception of the following perimeter, or next to perimeter states: (i) For $J = 0$ and n any of the allowed values then $\alpha = 1$; (ii) for $q + J + n =$ even, if $J + n = q + 2$ then $\alpha = 2$ and if $J - n = q + 3$ then $\alpha = 1$ and (iii) for $q + J + n =$ odd, if $J - n = q + 2$ then $\alpha = 2$ and if $J + n = q + 3$ then $\alpha = 1$.

Table 5. $\langle\langle q+3, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+3, q\rangle\alpha, n, J\rangle^{a,b}$.

α'	α	k	$q+J+n=$ odd
1	1	1	$\left[\frac{(2J+1)(2J+5)(q-J-n+3)(q+J+n+6)G(q,-n,J)}{4(J+1)G(q,-n-1,J+1)}\right]^{\frac{1}{2}}$
1	2	1	0
2	1	1	$-\left[\frac{12(q+2)(q+4)(q-J-n+1)(q-J-n+3)(q-J+n+2)(q+J-n+3)}{(J+1)G(q,-n-1,J+1)G(q,-n,J)}\right]^{\frac{1}{2}}$
2	2	1	$\left[\frac{(2J-1)(2J+3)(q-J-n+1)(q+J+n+4)G(q,-n-1,J+1)}{4(J+1)G(q,-n,J)}\right]^{\frac{1}{2}}$
1	1	−1	$\left[\frac{(2J-1)(2J+3)(q-J+n+2)(q+J-n+3)G(q,-n-1,J-1)}{4JG(q,-n,J)}\right]^{\frac{1}{2}}$
1	2	−1	$-\left[\frac{12(q+2)(q+4)(q-J-n+3)(q+J+n+4)(q+3J-n+3)^2}{JG(q,-n-1,J-1)G(q,-n,J)}\right]^{\frac{1}{2}}$
2	1	−1	0
2	2	−1	$\left[\frac{(2J-3)(2J+1)(q-J+n+4)(q+J-n+1)G(q,-n,J)}{4JG(q,-n-1,J-1)}\right]^{\frac{1}{2}}$
1	1	0	$-\mathrm{i}\left[\frac{9(2J+1)(q-J-n+1)(q-J-n+3)(q-J+n+2)(q-J+n+4)(2q+2J+7)^2}{4J(J+1)G(q,n+1,J)G(q,-n,J)}\right]^{\frac{1}{2}}$
1	2	0	$\mathrm{i}\left[\frac{3(2J-1)(2J+1)(2J+3)(q+2)(q+4)(q-J-n+1)(q-J+n+4)(q+J-n+3)(q+J+n+4)}{J(J+1)G(q,n+1,J)G(q,-n,J)}\right]^{\frac{1}{2}}$
2	1	0	$\mathrm{i}\left[\frac{3(2J-1)(2J+1)(2J+3)(q+2)(q+4)(q-J-n+3)(q-J+n+2)(q+J-n+3)(q+J+n+4)}{J(J+1)G(q,n+1,J)G(q,-n,J)}\right]^{\frac{1}{2}}$
2	2	0	$\mathrm{i}\left[\frac{(2J+1)G_1^2(q,n,J)}{4J(J+1)G(q,n+1,J)G(q,-n,J)}\right]^{\frac{1}{2}}$

[a] $G(q,n,J)$ as in table 4.
[b] $G_1(q,n,J) = (q+2)(14q^2+101q+183) - (14q+55)n(n+1) + \{8q^3+94q^2+350q+415+2(4q+17)[J^2 - n(n+1)] + (16q^2+114q+209)J\}J$.

Series $\langle q+4, q\rangle$. In this example the parameters take the values $J = 0, 1, \ldots, q+2$ and $n = (q-J+4), (q-j+4)-1, \ldots, -(q-J+4)$ whenever $J \neq 0$ or 1. If $J=0$ then $n = q, q-2, \ldots, -q$ and if $J=1$ then $n = q+1, q, \ldots, -(q+1)$.

The basis vectors of the series $\langle q+4, q\rangle$ with parameters satisfying $q+J+n=$ even, have in general multiplicities equal to three, and the corresponding matrix elements are presented in table 6. The following exceptions apply: (i) if $J \pm n = q+4$ or if $J = q+2$ and $n=0$ then $\alpha = 1$; (ii) if $J=1$ and $n = \pm(q+1)$ then $\alpha = 2$; (iii) if $J=0$ and n takes any allowed value then $\alpha = 3$; (iv) if $J \pm n = q+2$ then $\alpha = 1$ and 2 and (v) if $J=1$ and $n = q-1, q-3, \ldots, -(q-1)$ then $\alpha = 2$ and 3.

The matrix elements of the operator $\mathbf{U}$ applied to states with $q+J+n=$ odd are given in table 7. The multiplicity of the basis vectors is two in general, with the following exceptions: (i) if $J \pm n = q+3$ then $\alpha = 1$ and (ii) if $J = 0$ and n takes any of the allowed values then $\alpha = 2$.

In tables 6 and 7 the following expressions are introduced:

$$
\begin{aligned}
W(q,n,J) &= (q+3)(q+4)(q+5) - (q+2)n^2 - (q+5)J(J+1) \\
X(q,n,J) &= (q+3)(q+5)^2 - (q+3)n^2 - (q+6)J(J+1) \\
Y(q,n,J) &= 2(q+3)[(q+5)^2 - n^2][(q+3)(q+4)(q+5) - (q+2)n^2] \\
&\quad - [(q+5)(4q^3+46q^2+175q+222) - 6(q+3)n^2 \\
&\quad - (2q^2+16q+33)J(J+1)]J(J+1)
\end{aligned}
$$

Table 6. $\langle\langle q+4, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+4, q\rangle\alpha, n, J\rangle$.

α'	α	k	$q+J+n=$ even
1	1	1	$\left[\frac{(J-1)(J+3)(q-J-n+4)(q+J+n+7)Y_1(q,n,J)^2}{(j+1)Y(q,n+1,J+1)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	2	1	$-\left[\frac{2(J+2)(J+3)(q+3)(q+5)(q-J+n+4)(q+J-n+5)(q+J+n+5)(q+J+n+7)Z(q,n,J)}{(J+1)Y(q,n+1,J+1)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	3	1	0
2	1	1	$-\left[\frac{2J(J-1)(q+3)(q+5)(q-J-n+2)(q-J-n+4)(q-J+n+4)(q+J-n+5)Z(q,n+1,J+1)}{(J+1)Y(q,n+1,J+1)Y(q,n,J)}\right]^{\frac{1}{2}}$
2	2	1	$\left[\frac{J(J+2)(q-J-n+2)(q+J+n+5)Y_2(q,n,J)^2}{(J+1)Y(q,n+1,J+1)Z(q,n+1,J+1)Y(q,n,J)Z(q,n,J)}\right]^{\frac{1}{2}}$
2	3	1	$-\left[\frac{6(J+2)(q+2)(q+4)(q-J+n+2)(q+J-n+3)(q+J+n+3)(q+J+n+5)Y(q,n+1,J+1)}{Z(q,n+1,J+1)Z(q,n,J)}\right]^{\frac{1}{2}}$
3	1	1	0
3	2	1	$-\left[\frac{6J(q+2)(q+4)(q-J-n)(q-J-n+2)(q-J+n+2)(q+J-n+3)Y(q,n,J)}{Z(q,n+1,J+1)Z(q,n,J)}\right]^{\frac{1}{2}}$
3	3	1	$\left[\frac{(J-1)(q-J-n)(q+J+n+3)[Y_1(q,n,J)+2(2q+3)(2q+7)n(n+1)J(J+2)]^2}{Z(q,n+1,J+1)Z(q,n,J)}\right]^{\frac{1}{2}}$
1	1	−1	$\left[\frac{(J-2)(J+2)(q-J+n+6)(q+J-n+5)Z_1(q,n,J)^2}{JY(q,n+1,J-1)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	2	−1	$\left[\frac{2(J-2)(J-1)(q+3)(q+5)(q-J-n+4)(q-J+n+4)(q-J+n+6)(q+J+n+5)Z(q,n,J)}{JY(q,n+1,J-1)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	3	−1	0
2	1	−1	$\left[\frac{2(J+1)(J+2)(q+3)(q+5)(q-J-n+4)(q+J-n+3)(q+J-n+5)(q+J+n+5)Z(q,n+1,J-1)}{JY(q,n+1,J-1)Y(q,n,J)}\right]^{\frac{1}{2}}$
2	2	−1	$\left[\frac{(J-1)(J+1)(q-J+n+4)(q+J-n+3)Z_2(q,n,J)^2}{JY(q,n+1,J-1)Z(q,n+1,J-1)Y(q,n,J)Z(q,n,J)}\right]^{\frac{1}{2}}$
2	3	−1	$\left[\frac{6(J-1)(q+2)(q+4)(q-J-n+2)(q-J+n+2)(q-J+n+4)(q+J+n+3)Y(q,n+1,J-1)}{Z(q,n+1,J-1)Z(q,n,J)}\right]^{\frac{1}{2}}$
3	1	−1	0
3	2	−1	$\left[\frac{6(J+1)(q+2)(q+4)(q-J-n+2)(q+J-n+1)(q+J-n+3)(q+J+n+3)Y(q,n,J)}{Z(q,n+1,J-1)Z(q,n,J)}\right]^{\frac{1}{2}}$
3	3	−1	$\left[\frac{J(q-J+n+2)(q+J-n+1)[Z_1(q,n,J)+2(2q+3)(2q+7)n(n+1)(J-1)(J+1)]^2}{Z(q,n+1,J-1)Z(q,n,J)}\right]^{\frac{1}{2}}$
1	1	0	$-\left[\frac{2(2J+1)(q+3)(q-J-n+4)(q+J-n+5)[2(q+n+4)W(q,n+1,J)+(J-1)(J-2)(n+1)(2q+7)]^2}{J(J+1)W(q,n+1,J)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	2	0	$-\left[\frac{(J-1)(J+2)(2J+1)(q+5)(q-J+n+4)(q+J+n+5)Z(q,n,J)}{J(J+1)W(q,n+1,J)Y(q,n,J)}\right]^{\frac{1}{2}}$
1	3	0	0
2	1	0	$\left[\frac{2(J-1)(J+2)(2J+1)(q+2)(q+3)(q+5)(q-J-n+2)(q+J-n+3)[(q-J+4)^2-n^2][(q+J+5)^2-n^2]}{J(J+1)W(q,n+1,J)Y(q,n,J)}\right]^{\frac{1}{2}}$
2	2	0	$-\left[\frac{(2J+1)(q+2)(q-J-n+2)(q+J-n+3)[(q+n+4)Y(q,n,J)-2n(J-1)(J+2)(2q+7)X(q,n,J)]^2}{J(J+1)W(q,n+1,J)Y(q,n,J)Z(q,n,J)}\right]^{\frac{1}{2}}$
2	3	0	$-\left[\frac{6(2J+1)(q+4)(q-J+n+2)(q+J+n+3)W(q,n+1,J)}{Z(q,n,J)}\right]^{\frac{1}{2}}$

$$
\begin{aligned}
Z(q,n,J) &= Y(q,n,J) - 2(2q+7)\{2(q+3)^2[(q+5)^2-n^2] \\
&\quad - [2q^2+16q+33+(2q+3)n^2]J(J+1)\} \\
W_1(q,n,J) &= (q-n+3)Y(q,n+1,J) + 2(2q+7)(n+1)(J-1)(J+2)X(q,n+1,J) \\
Y_1(q,n,J) &= 2(q+2)(q+3)(q-n+3)(q-n+5)(q+n+5)^2 - 2(q+3)[(q+2)(2q+9) \\
&\quad - 3(n+1)(J+2)]nJ - (q+5)(2q+7)(4q^2+29q+48)J \\
&\quad - (4q^4+62q^3+347q^2+823q+681)J^2 + (2q^2+16q+33)(J+4)J^3
\end{aligned}
$$

Table 7. $\langle\langle q+4, q\rangle\alpha', n+1, J+k\|\mathbf{U}\|\langle q+4, q\rangle\alpha, n, J\rangle$.

α'	α	k	$q+J+n=\text{odd}$
1	1	1	$\left[\frac{(J-1)(J+3)(q-J-n+3)(q+J+n+6)[(q+2)(q+4)(q+5)-(q+2)n(n+1)-(q+5)J(J+2)]^2}{(J+1)W(q,n+1,J+1)W(q,n,J)}\right]^{\frac{1}{2}}$
1	2	1	$-\left[\frac{(J+2)(J+3)(q+2)(q+5)(q-J+n+3)(q+J-n+4)(q+J+n+4)(q+J+n+6)}{(J+1)W(q,n+1,J+1)W(q,n,J)}\right]^{\frac{1}{2}}$
2	1	1	$-\left[\frac{J(J-1)(q+2)(q+5)(q-J-n+1)(q-J-n+3)(q-J+n+3)(q+J-n+4)}{(J+1)W(q,n+1,J+1)W(q,n,J)}\right]^{\frac{1}{2}}$
2	2	1	$\left[\frac{J(J+2)(q-J-n+1)(q+J+n+4)[(q+3)(q+5)^2-(q+2)n(n+1)-(q+5)J(J+2)]^2}{(J+1)W(q,n+1,J+1)W(q,n,J)}\right]^{\frac{1}{2}}$
1	1	−1	$\left[\frac{(J-2)(J+2)(q-J+n+5)(q+J-n+4)[(q+3)^2(q+5)-(q+2)n(n+1)-(q+5)J^2]^2}{JW(q,n+1,J-1)W(q,n,J)}\right]^{\frac{1}{2}}$
1	2	−1	$\left[\frac{(J-2)(J-1)(q+2)(q+5)(q-J-n+3)(q-J+n+3)(q-J+n+5)(q+J+n+4)}{JW(q,n+1,J-1)W(q,n,J)}\right]^{\frac{1}{2}}$
2	1	−1	$\left[\frac{(J+1)(J+2)(q+2)(q+5)(q-J-n+3)(q+J-n+2)(q+J-n+4)(q+J+n+4)}{JW(q,n+1,J-1)W(q,n,J)}\right]^{\frac{1}{2}}$
2	2	−1	$\left[\frac{(J-1)(J+1)(q-J+n+3)(q+J-n+2)[(q+4)^2(q+5)-(q+2)n(n+1)-(q+5)J^2]^2}{JW(q,n+1,J-1)W(q,n,J)}\right]^{\frac{1}{2}}$
1	1	0	$\left[\frac{2(2J+1)(q+3)(q-J+n+5)(q+J+n+6)[2(q-n+3)W(q,n,J)-(2q+7)n(J-1)(J+2)]^2}{J(J+1)Y(q,n+1,J)W(q,n,J)}\right]^{\frac{1}{2}}$
1	2	0	$\left[\frac{2(J-1)(J+2)(2J+1)(q+2)(q+3)(q+5)(q-J+n+5)(q+J+n+6)[(q-J+3)^2-n^2][(q+J+4)^2-n^2]}{J(J+1)Y(q,n+1,J)W(q,n,J)}\right]^{\frac{1}{2}}$
2	1	0	$-\left[\frac{(J-1)(J+2)(2J+1)(q+5)(q-J-n+3)(q+J-n+4)Z(q,n+1,J)}{J(J+1)Y(q,n+1,J)W(q,n,J)}\right]^{\frac{1}{2}}$
2	2	0	$\left[\frac{(2J+1)(q+2)(q-J+n+3)(q+J+n+4)W_1(q,n,J)^2}{J(J+1)Y(q,n+1,J)Z(q,n+1,J)W(q,n,J)}\right]^{\frac{1}{2}}$
3	1	0	0
3	2	0	$-\left[\frac{6(2J+1)(q+4)(q-J-n+1)(q+J-n+2)W(q,n,J)}{Z(q,n+1,J)}\right]^{\frac{1}{2}}$

$$
\begin{aligned}
Y_2(q,n,J) = {} & Y(q,n,j)Z(q,n+1,J+1) + 6(2q+7)(n+1)(J+1)[(q+2)(q+3) \\
& - (q-2J+2)n - (q+3)J]Y(q,n,J) - 2(2q+7)n(J-1)[(q+3)(q+5) \\
& - (q-2J+3)n - (q+4)J]Z(q,n+1,J+1) \\
& -2(2q+7)n(n+1)J[3(J+1)(2J+2n+5)Y(q,n,J) \\
& - (J-1)(2J+2n+1)Z(q,n+1,J-1)]/(q+J+n+5) \\
Z_1(q,n,J) = {} & Y_1(q,n,J-1) + 2(q+2)(q+3)(2q+9)(2n+1)J \\
Z_2(q,n,J) = {} & Y(q,n+1,J-1)Z(q,n,J) - 6(2q+7)nJ[q^2+7q+13+(q-2J+4)n \\
& - (q+5)J]Y(q,n+1,J-1) + 2(2q+7)(n+1)(J-2)[(q+4)(q+6) \\
& + (q-2J+5)n - (q+6)J]Z(q,n,J) \\
& - 2(2q+7)n(n+1)(J-1)[3J(2J-2n+1)Y(q,n+1,J-1) \\
& - (J-2)(2J-2n-3)Z(q,n,J)]/(q+J-n+3).
\end{aligned}
$$

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Appendix

The reduced matrix elements of operators **U** and **V** satisfy a set of equations that follows from the commutation relations (1). They are presented in this appendix. For convenience, the non-vanishing matrix elements will be described by the matrices $\mathbf{PU}(\tau, j)$, $\mathbf{OU}(\tau, j)$, $\mathbf{MU}(\tau, j)$ and $\mathbf{PV}(\tau, j)$, $\mathbf{OV}(\tau, j)$, $\mathbf{MV}(\tau, j)$, defined as follows:

$$PU(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau + 1 j + 1 \, \|\mathbf{U}\| \, \alpha\tau j\rangle \qquad PV(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau - 1 j + 1 \, \|\mathbf{V}\| \, \alpha\tau j\rangle \tag{A.1}$$

$$OU(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau + 1 j \, \|\mathbf{U}\| \, \alpha\tau j\rangle \qquad OV(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau - 1 j \, \|\mathbf{V}\| \, \alpha\tau j\rangle \tag{A.2}$$

$$MU(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau + 1 j - 1 \, \|\mathbf{U}\| \, \alpha\tau j\rangle \qquad MV(\tau, j)_{\alpha'\alpha} = \langle\alpha'\tau - 1 j - 1 \, \|\mathbf{V}\| \, \alpha\tau j\rangle. \tag{A.3}$$

It is also convenient to introduce the matrix products

$$\mathbf{X}_1(\tau, j) = \mathbf{PU}(\tau, j)^+\mathbf{PU}(\tau, j) \qquad \mathbf{X}_4(\tau, j) = \mathbf{PV}(\tau, j)^+\mathbf{PV}(\tau, j) \tag{A.4}$$

$$\mathbf{X}_2(\tau, j) = \mathbf{OU}(\tau, j)^+\mathbf{OU}(\tau, j) \qquad \mathbf{X}_5(\tau, j) = \mathbf{OV}(\tau, j)^+\mathbf{OV}(\tau, j) \tag{A.5}$$

$$\mathbf{X}_3(\tau, j) = \mathbf{MU}(\tau, j)^+\mathbf{MU}(\tau, j) \qquad \mathbf{X}_6(\tau, j) = \mathbf{MV}(\tau, j)^+\mathbf{MV}(\tau, j). \tag{A.6}$$

The action of commutation relations (1) on a generic state $|\alpha\tau JM\rangle$, with the use of (2), leads to the following set of equations for the unknown reduced matrix elements:

$$\mathbf{PU}(\tau - 1, j + 1)\mathbf{PV}(\tau, j) = \mathbf{PV}(\tau + 1, j + 1)\mathbf{PU}(\tau, j) \tag{A.7}$$

$$\sqrt{2j+1}[(j + 1)\mathbf{OV}(\tau + 1, j + 1)\mathbf{PU}(\tau, j) + \mathbf{OU}(\tau - 1, j + 1)\mathbf{PV}(\tau, j)] = \sqrt{j(j+2)(2j+3)}\mathbf{PU}(\tau - 1, j)\mathbf{OV}(\tau, j) \tag{A.8}$$

$$\sqrt{2j+1}[\mathbf{OV}(\tau + 1, j + 1)\mathbf{PU}(\tau, j) + (j + 1)\mathbf{OU}(\tau - 1, j + 1)\mathbf{PV}(\tau, j)] = \sqrt{j(j+2)(2j+3)}\mathbf{PV}(\tau + 1, j)\mathbf{OU}(\tau, j) \tag{A.9}$$

$$\mathbf{X}_1(\tau, j) + \mathbf{X}_2(\tau, j) + \mathbf{X}_3(\tau, j) = (2j + 1)[C_2 - j(j + 1) - \tau^2 - 3\tau]\mathbf{I} \tag{A.10}$$

$$\mathbf{X}_4(\tau, j) + \mathbf{X}_5(\tau, j) + \mathbf{X}_6(\tau, j) = (2j + 1)[C_2 - j(j + 1) - \tau^2 + 3\tau]\mathbf{I} \tag{A.11}$$

$$(2j + 1)[\mathbf{X}_1(\tau, j) - \mathbf{X}_6(\tau, j)] + j(2j + 3)[\mathbf{X}_2(\tau, j) - \mathbf{X}_5(\tau, j)] = -\tau\lambda\mathbf{I} \tag{A.12}$$

$$(j + 1)(2j + 1)\mathbf{X}_1(\tau, j) - (2j + 3)[(j + 1)\mathbf{X}_4(\tau, j) + \mathbf{X}_5(\tau, j)] - \mathbf{X}_6(\tau, j) = -(j + \tau)\lambda\mathbf{I}. \tag{A.13}$$

Here, $C_2 = [(2q + p)(2q + p + 6) + p(p + 2)]/4$, $\lambda = (2j + 1)(2j + 2)(2j + 3)$ and **I** is the unit matrix of appropriate dimension.

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